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Bond percolation on subsets of the square lattice, and the threshold between one-dimensional and two-dimensional behaviour

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Abstract. For each value of $\pi \in [\frac{1}{2}, 1]$, we construct a subgraph $R(\pi)$ of the square lattice $\mathbb{Z}^2$ such that the bond percolation process on $R(\pi)$ has critical probability π. We show that the critical probability $\pi(a)$ of the region $\{(x, y): 0 \leq y \leq a \ln(x+1), 0 \leq x < \infty\}$ depends in a non-trivial way upon the choice of the number a; $\pi(a)$ is a continuous, decreasing function of a and satisfies $\pi(a) \to \frac{1}{2}$ as $a \to \infty$ and $\pi(a) \to 1$ as $a \downarrow 0$. We also construct a family of 'one-dimensional' bond percolation processes, whose critical probabilities range over the interval $(0, 1)$. All proofs are rigorous.

Let G be an infinite graph and let $\pi(G)$ be the critical probability of the bond percolation process on G. There is a small collection of graphs G for which $\pi(G)$ is known exactly. Here are two examples. The critical probability of a regular branching tree of degree $k+1$ equals k^{-1}; the critical probability of the square lattice $\mathbb{Z}^2$ equals $\frac{1}{2}$. In a recent paper, van den Berg (1982) posed the following question: is it true that for every prescribed $\pi \in [0, 1]$ there exists a graph G with critical probability π? He has answered this question in the affirmative, but for *no* value of $\pi (\notin \{0, \frac{1}{2}, 1\})$ was he able to construct explicitly the corresponding graph G. He used a probabilistic method which is based upon the following observation. Consider bond percolation on the square lattice $\mathbb{Z}^2$, each edge of which is declared *open* (respectively *closed*) with probability p (respectively $q = 1-p$). If $p > \frac{1}{2}$, then there exists almost surely (AS) an infinite open cluster; this cluster is a random subgraph of $\mathbb{Z}^2$ and has critical probability $(2p)^{-1}$. Hence the class of subgraphs of $\mathbb{Z}^2$ with critical probability $(2p)^{-1}$ is not empty for any p satisfying $\frac{1}{2} < p \leq 1$. It is the purpose of this paper to indicate that an alternative analytical approach yields explicit constructions of subgraphs of $\mathbb{Z}^2$ with critical probabilities ranging over the interval $[\frac{1}{2}, 1]$.

Consider bond percolation on $\mathbb{Z}^2$, in which each edge is open with probability p; we write $P_p(A)$ for the probability of an event A. Let f: $[0, \infty) \to [0, \infty)$ be a function on the interval from zero to infinity, taking non-negative values, and define $\mathbb{Z}^2(f)$ to be the region of $\mathbb{Z}^2$ which is bounded by the lines $y = f(x)$ and $y = 0$; thus

$$\mathbb{Z}^2(f) = \{(x, y) \in \mathbb{Z}^2: 0 \leq y \leq f(x), 0 \leq x < \infty\}.$$

Let $I(f, \infty)$ be the event that $\mathbb{Z}^2(f)$ contains an infinite open cluster; $I(f, \infty)$ is in the trivial tail σ-field of a collection of independent Bernoulli random variables (see Grimmett and Stirzaker 1982, pp 21, 29), and so $P_p(I(f, \infty))$ equals either 0 or 1

(Grimmett and Stirzaker 1982, p 190). The *critical probability* $\pi(\mathbb{Z}^2(f))$ is defined as

$$\pi(\mathbb{Z}^2(f)) = \sup\{p : P_p(I(f, \infty)) = 0\}.$$

π is the unique number with the property that if $p > \pi$ then there exists (AS) an infinite open cluster in that part of the (x, y) plane bounded by the x axis and the line $y = f(x)$, whilst if $p < \pi$ then (AS) no such cluster exists. We shall prove the following theorem.

Theorem. If $f(x) = a \ln(x+1)$ where $0 < a < \infty$ then the critical probability $\pi(a)$ of $\mathbb{Z}^2(f)$ is a function $\pi: (0, \infty) \to (\frac{1}{2}, 1)$ with the properties

$$\pi(a) \text{ is a continuous function of } a \tag{1}$$

$$\pi(a) \text{ is a strictly decreasing function of } a \tag{2}$$

$$\pi(a) \uparrow 1 \text{ as } a \downarrow 0, \text{ and } \pi(a) \downarrow \tfrac{1}{2} \text{ as } a \uparrow \infty. \tag{3}$$

This theorem has the following immediate consequences.

(i) For any $\pi \in (\frac{1}{2}, 1)$, there exists $a \in (0, \infty)$ such that the region of $\mathbb{Z}^2$ between the lines $y = a \ln(x+1)$ and $y = 0$ has critical probability π.

(ii) If $f: [0, \infty) \to [0, \infty)$ and $f(x)/\ln x \to 0$ as $x \to \infty$ then $\mathbb{Z}^2(f)$ has critical probability 1. Roughly speaking, such regions behave as one-dimensional regions.

(iii) If $f: [0, \infty) \to [0, \infty)$ and $f(x)/\ln x \to \infty$ as $x \to \infty$ then $\mathbb{Z}^2(f)$ has critical probability $\frac{1}{2}$. Roughly speaking, such regions behave as the whole lattice $\mathbb{Z}^2$.

(iv) The device of van den Berg (1982) may be applied to construct graphs with critical probabilities ranging over the interval $(0, 1]$; this involves replacing each edge of $\mathbb{Z}^2$ by some fixed number, m say, of parallel edges.

The theorem is proved by refinements of the methods of Grimmett (1981). The choice of the square lattice as the underlying graph is convenient, since at least as much is known about this lattice as about any other non-trivial lattice, but similar methods should be valid for any two-dimensional lattice.

The above argument provides a truly two-dimensional construction of graphs with prescribed critical probabilities. Before proving the theorem, we indicate a simple way of constructing 'one-dimensional' graphs with prescribed critical probabilities. Fix $\pi \in [0, 1]$, and let $G(\pi)$ be the graph defined as follows. The vertices of $G(\pi)$ are labelled $1, 2, \ldots$; the edges of $G(\pi)$ are such that vertices k and j are adjacent if and only if $|k - j| = 1$, and for each $k \geq 2$, vertices $k-1$ and k are joined by $b(k)$ edges in parallel, where

$$b(k) = \begin{cases} k & \text{if } \pi = 0 \\ \operatorname{int}(c(k)) & \text{if } 0 < \pi < 1 \\ 1 & \text{if } \pi = 1 \end{cases}$$

and

$$c(k) = \frac{\ln k}{\ln(1/(1-\pi))}.$$

Here, $\operatorname{int}(x)$ denotes the integer part of x. It is an easy exercise to verify that $G(\pi)$ has critical probability π.

Proof of the theorem. A *path* in $\mathbb{Z}^2$ is an alternating sequence $\{v_0, e_0, v_1, \ldots, e_n, v_{n+1}\}$ of vertices and edges such that e_i joins v_i to v_{i+1} (for each i) and $v_i \neq v_j$ if $i \neq j$. A path is called *open* (respectively *closed*) if all its edges are open (respectively closed). Suppose that $\frac{1}{2} < p = 1 - q < 1$, so that $0 < q < \frac{1}{2}$.

Let L_k be the line $x = k$ in $\mathbb{Z}^2$, and let $\beta_q(k)$ be given by

$$\beta_q(k) = P(\mathbf{O} \text{ is joined to a vertex on } L_k \text{ by a closed path contained in } C_{0k}, \text{ except for the first vertex}),$$

where $\mathbf{O}$ is the origin of $\mathbb{Z}^2$ and

$$C_{mn} = \{(x, y) \in \mathbb{Z}^2 \colon m < x \leq n\}.$$

We recall some results from Grimmett (1981). There exists a number α_q such that $0 < \alpha_q < \infty$ and

$$k^{-1} \ln \beta_q(k) \to -\alpha_q \text{ as } k \to \infty. \tag{4}$$

Furthermore if $f(x) = a \ln(x+1)$ and $p + q = 1$ then

$$P_p(I(f, \infty)) = \begin{cases} 0 & \text{if } a\alpha_q < 1 \\ 1 & \text{if } a\alpha_q > 1. \end{cases} \tag{5}$$

In order to prove the theorem, it is sufficient to show that α_q is a continuous, strictly decreasing function of q (and therefore a continuous, strictly increasing function of $p = 1 - q$) which maps $(0, \frac{1}{2})$ into $(0, \infty)$, and that

$$\alpha_q \to \infty \text{ as } q \downarrow 0 \qquad \text{and} \qquad \alpha_q \to 0 \text{ as } q \uparrow \tfrac{1}{2}. \tag{6}$$

To see that this is sufficient, suppose that it holds and define ν to be the inverse function of α_q: ν is given uniquely by

$$\alpha_{\nu(x)} = x$$

and is a continuous, strictly decreasing function which maps $(0, \infty)$ onto $(0, \frac{1}{2})$. It is an immediate consequence of (5) that

$$\pi(a) = 1 - \nu(a^{-1}).$$

The remainder of the proof is devoted to proving the above remarks about α_q. It is divided into three lemmas.

Lemma 1. α_q is a continuous function of q on $(0, \frac{1}{2})$.

Proof. In Grimmett (1981), it was shown that α_q is a lower semicontinuous function of q; this followed from the fact that α_q is the limit of a superadditive sequence of continuous functions of q, and thus may be approximated from below, to any prescribed degree of accuracy, by a continuous function. To show upper semicontinuity, we observe first that

$$\beta_q(r + s) \geq \beta_q(r)\beta_q(s) \qquad \text{for } r, s \geq 1. \tag{7}$$

To see this, we order the vertices on the line $x = k$ in some fixed manner $v_1, v_2, \ldots$ and define

$$K = \min\{k \colon \mathbf{O} \text{ is joined to } v_k \text{ by a closed path contained in } C_{0k}, \text{ except for the first vertex}\}$$

with the convention that $K = \infty$ if no such path exists. Then

$$\beta_q(r+s) \geq \sum_{k=1}^{\infty} P(K=k, \text{ and } v_k \text{ is joined to } L_{r+s} \text{ by a closed path in } C_{r,r+s}, \text{ except for the first vertex})$$

$$= \sum_{k=1}^{\infty} P(K=k)\beta_q(s) \qquad \text{by independence}$$

$$= \beta_q(r)\beta_q(s).$$

By the theory of subadditive functions, the limit α_q associated with the subadditive sequence $\{-\ln \beta_q(k): k = 1, 2, \ldots\}$ satisfies

$$\alpha_q = \inf_k \left(-k^{-1} \ln \beta_q(k)\right).$$

Thus α_q may be approximated from above, to any prescribed degree of accuracy, by one of the family $\{-k^{-1} \ln \beta_q(k): k = 1, 2, \ldots\}$, and we need only to verify that $\beta_q(k)$ is continuous for each k. To see this, note that

$$\beta_q(k, n) \leq \beta_q(k) \leq \beta_q(k, n) + \gamma_q(n)$$

where $\beta_q(k, n)$ is the probability that $\mathbf{O}$ is joined to L_K by a closed path in C_{0k} containing fewer than n edges, and $\gamma_q(n)$ is the probability that there exists a closed path starting from $\mathbf{O}$ with n or more edges. But $\gamma_q(n)\downarrow 0$ as $n \to \infty$, since $q \leq \frac{1}{2}$; by Dini's theorem , we have that $\gamma_q(n) \to 0$ uniformly on $q \in [0, \frac{1}{2}]$. Therefore $\beta_q(k, n) \to \beta_q(k)$ as $n \to \infty$, uniformly on $[0, \frac{1}{2}]$; but $\beta_q(k, n)$ is continuous in q, giving that $\beta_q(k)$ is continuous in q on $(0, \frac{1}{2})$. The proof of lemma 1 is now complete. Similar arguments may be used to show that the function π_p of Grimmett (1981) is continuous on $(0, \frac{1}{2})$.

Lemma 2. α_q is a strictly decreasing function of q on $(0, \frac{1}{2})$.

Proof. Let $0 < u < v < \frac{1}{2}$. It is clear that $\alpha_u \geq \alpha_v$. We construct both the percolation processes (with edge-probabilities u and v) on the same probability space in the following way. With each edge e we associate two independent Bernoulli random variables $X(e)$ and $Y(e)$, each of which may take the value 0 or 1, such that

$$P(X(e) = 1) = v \qquad P(Y(e) = 1) = u/v.$$

We colour e *black* if $X(e) = 0$, *white* if $X(e)Y(e) = 1$, and *grey* otherwise; we call e *light* if it is either white or grey. The white edges constitute a percolation process with edge-probability u and the light edges constitute a process with edge-probability v. Let V be the collection of edges in the light cluster which contains the origin $\mathbf{O}$ of $\mathbb{Z}^2$. It is a consequence of theorem 1 of Kesten (1981) that there exist constants $c_1, c_2 > 0$ such that

$$P(|V| > k) \leq c_1 \exp(-c_2 k) \qquad \text{for all } k. \tag{8}$$

Let M be a positive constant, to be chosen shortly, and write

$$\varepsilon = 1 - (u/v)$$

as the probability that a light edge is *not* white; note that $\varepsilon > 0$. For each $k = 1, 2, \ldots$, let $\mathscr{E}(k)$ be the collection of edges of $\mathbb{Z}^2$ which join two vertices of the form

$(k-1, y)$ and (k, y), for $y = 0, \pm 1, \ldots$; let $N(k)$ be the number of edges in $\mathscr{E}(k)$ which have the property of belonging to a light path beginning at $\mathbf{O}$ and lying entirely within $C_{0\infty}$ except for its first vertex, and write $\boldsymbol{N}(k)$ for the random k-vector $(N(1), \ldots, N(k))$. Let W_k be the event that $\mathbf{O}$ is joined to L_k by a white path contained in C_{0k}, except for its first vertex. Clearly, if $\boldsymbol{n}(k) = (n(1), \ldots, n(k))$, then

$$P(W_k|\boldsymbol{N}(k) = \boldsymbol{n}(k)) \leq (1-\varepsilon^{n(1)})(1-\varepsilon^{n(2)}) \ldots (1-\varepsilon^{n(k)}) \leq \exp\left(-\sum_{i=1}^{k} \varepsilon^{n(i)}\right) \tag{9}$$

since each of the light edges contributing to $N(i)$ is *not* white with probability ε. Now,

$$\beta_u(k) = P(W_k) = \sum_{\boldsymbol{n}(k)} P(W_k|\boldsymbol{N}(k) = \boldsymbol{n}(k)) P(\boldsymbol{N}(k) = \boldsymbol{n}(k))$$

where the sum is over all k-vectors $\boldsymbol{n}(k)$ with non-zero entries. We divide the sum into two parts, depending on whether $\Sigma_i n(i) \leq Mk$ or $\Sigma_i n(i) > Mk$, to obtain from (9) that

$$\beta_u(k) \leq \sum_{\boldsymbol{n}:\Sigma_i n(i) \leq Mk} \exp\left(-\sum_{i=1}^{k} \varepsilon^{n(i)}\right) P(\boldsymbol{N}(k) = \boldsymbol{n}(k)) + P(|V| > Mk).$$

By the arithmetic/geometric mean inequality,

$$\beta_u(k) \leq \sum_{\boldsymbol{n}:\Sigma_i n(i) \leq Mk} \exp(-k\varepsilon^M) P(\boldsymbol{N}(k) = \boldsymbol{n}(k)) + P(|V| > Mk)$$
$$\leq \exp(-k\varepsilon^M)\beta_v(k) + c_1 \exp(-c_2 Mk)$$

from (8). Choose M to be large enough so that

$$c_2 M > \varepsilon^M + 2\alpha_v$$

to deduce from (4) that

$$\beta_u(k) \leq 2\beta_v(k) \exp(-k\varepsilon^M) \qquad \text{for all large } k$$

and combine this with (4) to see that

$$\alpha_u \geq \alpha_v + \varepsilon^M.$$

The proof of lemma 2 is now complete.

Lemma 3. $\alpha_q \to \infty$ as $q \downarrow 0$, and $\alpha_q \to 0$ as $q \uparrow \frac{1}{2}$.

Proof. The first limit follows from the fact that

$$\beta_q(k) \leq \gamma_q(k) \leq q^k \kappa^{k+o(k)}$$

where κ is the connective constant of $\mathbb{Z}^2$, giving that

$$\alpha_q \geq -\ln(\kappa q) \to \infty \qquad \text{as } q \downarrow 0.$$

Finally, note that the proof of lemma 1 may be extended to show that α_q is upper semicontinuous on $(0, \frac{1}{2}]$; it follows that α_q is left-continuous at $q = \frac{1}{2}$, and the second limit follows from the observation (Grimmett 1981) that $\alpha_{1/2} = 0$. This proves lemma 3 and completes the proof of the theorem.

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